

A GALOIS SIDE ANALOGUE OF A THEOREM OF BERNSTEIN

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ABSTRACT. Let G be a connected reductive group defined over a non archimedean local field k . A theorem of Bernstein states that for any compact subgroup K of $G(k)$, there are, upto unramified twists, only finitely many K -spherical supercuspidal representations of $G(k)$. We prove an analogous result on the Galois side of the Langlands correspondence.

1. INTRODUCTION

Let G be a connected reductive group defined over a non-archimedean local field k . Let $X_{\text{nr}}(G(k))$ be the group of *unramified characters* of $G(k)$ (Definition 12). For a smooth representation π of $G(k)$, the various representations $\pi \otimes \chi$, $\chi \in X_{\text{nr}}(G(k))$ are called the unramified twists of π . A Theorem of Bernstein [Roc09, Theorem 1.4.2.1] states that

Theorem 1 (Bernstein). *For each compact open subgroup K of $G(k)$, the number of isomorphism classes, up to unramified twists, of irreducible cuspidal representations of $G(k)$ having non-zero K -fixed vectors is finite.*

On the other hand, local Langlands conjectures predict that “packets” of irreducible admissible representations of $G(k)$ should be parametrized by *Langlands parameters*, which are *admissible* elements of $H^1(W'_k, \hat{G})$ (Definition 4), where W'_k is the Weil-Deligne group and \hat{G} is the complex dual of G (see [Bor79]). Under this conjectural correspondence, supercuspidal representations are expected to correspond to *discrete Langlands parameters* (Definition 9). By Langlands philosophy, one should expect a result analogous to Theorem 1 on the parameter side.

Let W_k be the Weil group of k and I_k its inertia subgroup. Let Fr be a Frobenius element in W_k . Let J be an open subgroup of I_k which is normal in W_k . Call two Langlands parameters to be equivalent if they are in the same $H^1(W_k/I_k, (Z(\hat{G})^{I_k})^\circ)$ orbit (see (5.1)), where $Z(\hat{G})$ is the center of \hat{G} . In Theorem 10, we show that upto this equivalence, there are only finitely many discrete Langlands parameters which are trivial on J . In Section 7, using Kottwitz homomorphism, we observe that $H^1(W_k/I_k, (Z(\hat{G})^{I_k})^\circ)$ is isomorphic to the group $X_{\text{nr}}(G(k))$ of unramified characters of $G(k)$.

These statements are thus consistent with the conjectures in [Bor79, Section 10.3 (2)].

2. NOTATIONS

Let k be a non-archimedean local field and fix an algebraic closure \bar{k} of k . Let W_k denote the Weil group of k and I_k denote its inertia subgroup. We fix a Frobenius element Fr in W_k . For any algebraic group \mathcal{G} , we will denote by $Z(\mathcal{G})$ the center of \mathcal{G} . For any subgroup \mathcal{H} of \mathcal{G} , we will denote by $Z_{\mathcal{G}}\mathcal{H}$, the centralizer of \mathcal{H} in \mathcal{G} . The identity component of \mathcal{G} will be denoted by \mathcal{G}° .

3. REPRESENTATIONS OF THE WEIL GROUP

Let J be an open subgroup of I_k which is normal in W_k .

Definition 2. A representation of W_k is called unramified, if it is trivial on I_k .

Lemma 3. *Upto unramified twists, there exist only finitely many irreducible representations of W_k which are trivial on J .*

Proof. Let (ρ, V) be an irreducible representation of W_k such that $J \subset \ker \rho$. Let $\text{Fr} \in W_k$ be a Frobenius element in W_k . It acts by conjugation on the finite group I_k/J , so some power Fr^d , $d \geq 1$ acts trivially. Thus $\rho(\text{Fr}^d)$ commutes with $\rho(I_k)$ and since ρ is irreducible, $\rho(\text{Fr}^d)$ must be scalar by Schur's lemma. Let χ be an unramified character of W_k such that $\chi(\text{Fr}^d) = \rho(\text{Fr}^d)$. Thus ρ is of the form $\chi \otimes \tau$ where τ is an irreducible representation of the finite group $W_k/\langle \text{Fr}^d, J \rangle$. Thus there are upto unramified twists, only finitely many irreducible representations of W_k which are trivial on J . \square

4. LANGLANDS PARAMETERS

Let G be a connected, reductive group over k and let k_0 be the splitting field in \bar{k} of the quasi-split inner form of G . Let ${}^L G = \hat{G} \rtimes \text{Gal}(k_0/k)$, where \hat{G} is the complex dual of G . The center ${}^L Z$ of ${}^L G$ is the group of $\text{Gal}(k_0/k)$ -fixed points in the center of \hat{G} .

Definition 4. A homomorphism $\varphi : W_k \times \text{SL}(2, \mathbb{C}) \rightarrow {}^L G$ is called *admissible* if

- (1) $\varphi : \text{SL}(2, \mathbb{C}) \rightarrow \hat{G}$ is a homomorphism of algebraic groups over \mathbb{C} .
- (2) φ is continuous on I_k and $\varphi(\text{Fr})$ is semisimple.
- (3) The composite $W_k \xrightarrow{\varphi} {}^L G \rightarrow \text{Gal}(k_0/k)$ is the canonical surjection $W_k \rightarrow \text{Gal}(k_0/k)$.

Two admissible homomorphisms are equivalent if they are conjugate by \hat{G} . A *Langlands parameter* is an equivalence class of admissible homomorphisms.

The group $W'_k := W_k \times \mathrm{SL}(2, \mathbb{C})$ is called the Weil-Deligne group of k . It is sometimes more convenient to see a Langlands parameter as an element of $H^1(W'_k, \hat{G})$.

Definition 5. A Langlands parameter is *unramified* if it is trivial on I_k and $\mathrm{SL}(2, \mathbb{C})$.

5. FINITENESS RESULT

Let the notations be as in Section 4. So G is as before a connected reductive group defined over k . Let $\hat{\mathfrak{g}}$ be the Lie algebra of the complex dual \hat{G} of G . Let $W'_k := W_k \times \mathrm{SL}(2, \mathbb{C})$ denote the Weil-Deligne group of k . Let J be an open subgroup of I_k which is normal in W_k . Let $\Phi(G)$ denote the set of Langlands parameters of G .

We have a well defined action

$$(5.1) \quad H^1(W_k, Z(\hat{G})) \times \Phi(G) \rightarrow \Phi(G), \quad [\alpha] \cdot [\phi] \mapsto [\alpha \cdot \phi]$$

Definition 6. Call two parameters φ, φ' to be equivalent if they are in the same $H^1(W_k/I_k, (Z(\hat{G})^J)^\circ)$ orbit.

Lemma 7. Let T be a tori defined over k and let \hat{T} be its complex dual. There are only finitely many equivalence classes of Langlands parameters for T which are trivial on J .

Proof. We have a canonical decomposition $H^1((\langle \mathrm{Fr} \rangle \rtimes I_k)/J, \hat{T}) = H^1(\langle \mathrm{Fr} \rangle, (\hat{T}^{I_k})^\circ) \times H(I_k/J, \hat{T})$. Let $d_J = |I_k/J|$. Then any element of $H^1(I_k/J, \hat{T})$ is killed by d_J . Thus the image of these elements lies in the d_J -torsion points of \hat{T} which is a finite set. Therefore $H^1(I_k/J, \hat{T})$ is finite. \square

Lemma 8. Let $\varphi : W'_k \rightarrow {}^L G$ be an admissible homomorphism which is trivial on J . If $\mathrm{image}(\varphi)$ is not contained in any proper parabolic subgroup of ${}^L G$, then there exists a number $n = n(J, G)$ such that $\varphi(\mathrm{Fr}^n) \in Z({}^L G)$.

Proof. Let d be a positive integer such that Fr^d acts trivially on I_k/J . Then $\varphi(\mathrm{Fr}^d) \in Z(\mathrm{image}(\varphi))$. Let $l = |\mathrm{Gal}(k_0/k)|$. Then $s := \varphi(\mathrm{Fr}^d)^l \in \hat{G}$. Let $H = Z_{{}^L G}(s)$. Then $\mathrm{image}(\varphi) \subset H$ and $s \in Z(H)$. The group $Z_{{}^L G}(Z(H)^\circ)$ is a Levi subgroup of ${}^L G$ containing H and therefore must be ${}^L G$ since $\mathrm{image}(\varphi)$ is not contained in any proper parabolic subgroup. Thus $Z(H)^\circ \subset Z({}^L G)$. From the structure theorem of the centralizers of semisimple elements, we know that there can be only finitely many possibilities for H [Kur83, Prop. 2.1]. Since $s \in Z(H)$, the fact that there are only finitely many possibilities for H allows us to choose a positive integer $a = a(G)$ independent of H such that $s^a \in Z(H)^\circ$. The Lemma follows. \square

Definition 9. A Langlands parameter is called discrete if its image is not contained in any parabolic subgroup of ${}^L G$.

Theorem 10. *Let G be a connected reductive group over k . Then there exist only finitely many equivalence classes of discrete Langlands parameters for G which are trivial on J .*

Proof. Let $\varphi : W'_k \rightarrow {}^L G$ be an admissible homomorphism. By Lemma 8, there exists an integer $n = n(J, G)$ such that the composite map $\bar{\varphi} : W'_k \rightarrow {}^L G \rightarrow \mathcal{G} := \hat{G}_{\text{ad}} \rtimes \text{Gal}(k_0/k)$ factors through $W_k / \langle \text{Fr}^n, J \rangle \times \text{SL}(2, \mathbb{C})$. By [Slo97, II.3, Theorem 1], there are only finitely many \mathcal{G} conjugacy classes of homomorphisms $W_k / \langle \text{Fr}^n, J \rangle \rightarrow \mathcal{G}$. It follows that there are only finitely many \mathcal{G} conjugacy classes of homomorphisms $W'_k \rightarrow \mathcal{G}$ which are trivial on J .

Now if $\varphi_1, \varphi_2 \in H^1(W'_k, \hat{G})$ are two Langlands parameters such that their images in $H^1(W'_k, \hat{G}_{\text{ad}})$ are equal, then $\varphi_1 = \varphi_c \cdot \varphi_2$ where $\varphi_c \in H^1(W_k, Z(\hat{G}))$. By Lemma 7, there are only finitely many such φ_c upto equivalence. The theorem follows. \square

The Weil group W_k carries an upper numbering filtration $\{W_k^r\}_{r \geq 0}$. The depth of a parameter $\varphi : W'_k \rightarrow {}^L G$ is defined to be

$$\inf\{r \geq 0 : W_k^s \subset \ker(\varphi) \text{ for } s > r\}.$$

Corollary 11. *There exist only finitely many equivalence classes of Langlands parameters of a given depth.*

6. UNRAMIFIED CHARACTERS

Let $X_k(G) = \text{Hom}(G, \mathbb{G}_m)$, the lattice of k -rational characters of G . Let

$$G(k)^1 = \{g \in G(k) : \text{val}_k(\chi(g)) = 0, \forall \chi \in X_k(G)\}.$$

Then $G(k)^1$ is an open normal subgroup of $G(k)$ that contains each compact subgroup of $G(k)$. It also has the following properties:

- (1) $G(k)^1$ has compact center;
- (2) $G(k)/G(k)^1$ is a free abelian group of finite rank;
- (3) The center $Z(G(k)^1)$ of $G(k)^1$ has finite rank in $G(k)$.

Definition 12. The group $X_{\text{nr}}(G(k))$ of *unramified characters* of $G(k)$ is defined by

$$X_{\text{nr}}(G(k)) = \text{Hom}(G(k)/G(k)^1, \mathbb{C}^\times).$$

Definition 13. For a smooth representation π of $G(k)$, the representations $\pi \otimes \chi$, $\chi \in X_{\text{nr}}(G(k))$ are called the *unramified twists* of π .

Theorem 14 (Bernstein). *For each compact open subgroup K of $G(k)$, the number of isomorphism classes, up to unramified twists, of irreducible cuspidal representations τ of $G(k)$ with $\tau^K \neq 0$ is finite.*

7. LANGLANDS PARAMETERS FOR UNRAMIFIED CHARACTERS

In [Kot97, Section 7], Kottwitz defined a surjective homomorphism

$$\kappa_G : G(k) \rightarrow X^*(Z(\hat{G}))_{I_k}^{\text{Fr}}.$$

Let $v_G : G(k) \rightarrow X^*(Z(\hat{G}))_{I_k}^{\text{Fr}} / \text{torsion}$ be the homomorphism induced by the Kottwitz homomorphism. Then $\ker v_G = G(k)^1$ (see [HR08, Remark 10]). We therefore have:

$$\begin{aligned} X_{\text{nr}}(G(k)) &\cong \text{Hom}(X^*(Z(\hat{G}))_{I_k}^{\text{Fr}} / \text{torsion}, \mathbb{C}^\times) \\ &\cong \text{Hom}(X^*((Z(\hat{G})^{I_k})^\circ_{\text{Fr}}), \mathbb{C}^\times) \\ (7.1) \quad &\cong (Z(\hat{G})^{I_k})^\circ_{\text{Fr}}. \end{aligned}$$

The last equality holds by Cartier duality.

We have

$$(7.2) \quad (Z(\hat{G})^{I_k})^\circ_{\text{Fr}} \cong H^1(W_k/I_k, (Z(\hat{G})^{I_k})^\circ) \hookrightarrow H^1(W_k, Z(\hat{G})).$$

Combining equations (7.1) and (7.2), we get a map

$$(7.3) \quad X_{\text{nr}}(G(k)) \hookrightarrow H^1(W_k, Z(\hat{G})) \rightarrow H^1(W'_k, \hat{G}).$$

One can thus associate to the unramified characters, Langlands parameters whose image lie in the center of \hat{G} .

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